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THE INDIVIDUAL ERGODIC THEOREM FOR p-SEQUENCES

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ABSTRACT

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let T be a measure-preserving weak mixing transformation. We define a large class of sequences of integers called p-sequences, such that if $f \in L_1$ there exists a set $\Omega' \subset \Omega$ of probability one and for $\omega \in \Omega'$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(T^{k_j}\omega) = \int_{\Omega} f(\omega)\mu(d\omega)$$

for every p-sequence $\{k_n\}$.

Preliminaries

Let G be a locally compact abelian group and let $\hat{G}$ be its dual, i.e. $\hat{G}$ is the collection of continuous homomorphisms from G to the complex unit circle C, endowed with the topology inherited from G. Let $\hat{G}_d$ be $\hat{G}$ with the discrete topology. Then $\hat{\hat{G}}_d = \bar{G}$ is called the Bohr compactification of G. $\bar{G}$ is a compact group such that G is a dense subset of $\bar{G}$ and with m being Haar measure on $\bar{G}$ we have $m(G) = 0$. Now let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of probability measures defined on the Borel sets of G. In an obvious way we may consider each μ_n as a measure on the Borel sets of $\bar{G}$. We shall call such a sequence $\{\mu_n\}$ ergodic if $\lim_n \mu_n = m$ weakly, i.e. if for every continuous function f defined on $\bar{G}$ we have

$$\lim_n \int_{\bar{G}} f(g)\mu_n(dg) = \int_{\bar{G}} f(g)m(dg).$$

The terminology arises from the fact that for such sequences one has a generalized mean ergodic theorem. If $U(g)$ is a continuous unitary representation of G on some Hilbert space H, then strong $\lim_n \int_G U(g)\mu_n(dg) = P$, where P is the projection on the subspace $\{f \in H \mid U(g)f = f \text{ for all } g \in G\}$, as was

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shown in [1]. Now let $0 \leqq p \leqq 1$ and let S be a Borel subset of G. We shall call S a p-set if $\lim_n \int_G \chi_S(g)\mu_n(dg) = P$, for every ergodic sequence $\{\mu_n\}$. While we have been unable to characterize p-sets completely it is very easy to provide examples of them. Let A be a Borel subset of $\bar{G}$ such that $m(\partial A) = 0$, i.e. A is a continuity set for Haar measure. Let $S = \bar{A} \cap G$. Then it follows at once that $\lim_n \int_G \chi_S(g)\mu_n(dg) = m(A)$ for every ergodic sequence $\{\mu_n\}$. Thus S is a p-set with $p = m(A)$. To construct such a set let B be a continuity set for Lebesgue measure on C, and let $A = \{g \in \bar{G} \mid \langle \gamma, g\rangle \in B\}$, where γ is an arbitrary character of $\bar{G}$ of infinite order. Then it is not difficult to show that A is a continuity set for m and that $m(A) = l(B)$, where l is Lebesgue measure on C. Finally it is again easy to see that $S = \{g \in G \mid \langle \gamma, g\rangle \in B\}$. When $G = Z$ and α is an irrational multiple of π we obtain in this way $S = \{n \in Z \mid e^{in\alpha} \in B\}$. Thus S is a p-set with $p = l(B)$.

Ergodic theorems

Now let $(\Omega, \mathcal{F}, P)$ be a probability space and let T be a measure-preserving transformation of Ω into itself. T is ergodic if

$$\lim_n \frac{1}{n} \sum_{j=1}^{n} P(T^j A \cap B) = P(A)P(B)$$

for all $A, B \in \mathcal{F}$, and T is weakly mixing if

$$\lim_n \frac{1}{n} \sum_{j=1}^{n} |P(T^j A \cap B) - P(A)P(B)| = 0.$$

We shall use a remarkable theorem of Wiener and Wintner [4] which we state here.

THEOREM (Wiener–Wintner). *Let T be ergodic and let $f \in L_1(P)$. Then there exists a set Ω' of probability one such that for $\omega \in \Omega'$ we have $\lim_{n\to\infty}(1/n)\sum_{j=1}^n f(T^j\omega)e^{ij\alpha}$ exists for all α with $0 \leqq \alpha \leqq 2\pi$. If moreover T is weakly mixing then the above limit is zero for $0 < \alpha < 2\pi$.*

The fact that for each fixed α there exists a set of probability one for which the limit exists is an immediate consequence of the individual ergodic theorem. The interesting feature of the Wiener–Wintner theorem is that the set of convergence is independent of α.

Now suppose $f \in L_1(P)$. Without loss of generality assume that $f \geqq 0$ and that $\int_\Omega f(\omega)P(d\omega) = 1$. Suppose T is weakly mixing. It follows from the above theorem that $(1/n)\sum_{j=1}^n f(T^j\omega)e^{ij\alpha} \to 0$ for $0 < \alpha < 2\pi$ simultaneously for almost every ω. Pick such an ω and define the sequence of measures $\{\mu_{n,\omega}\}$ on Z by $\mu_{n,\omega}(j) = f(T^j\omega)/n$, for $j = 1, \cdots, n$ and $\mu_{n,\omega}(j) = 0$ otherwise. Then the Fourier transform $\hat{\mu}_{n,\omega}(\alpha) = (1/n)\Sigma_{j=1}^n f(T^j\omega)e^{ij\alpha}$ and it follows from the above that

$$\lim_{n\to\infty} \hat{\mu}_{n,\omega}(\alpha) = \begin{cases} 0, & 0 < \alpha < 2\pi \\ 1, & \alpha = 0 \end{cases}.$$

But if m is Haar measure on $\bar{Z}$ then

$$\hat{m}(\alpha) = \begin{cases} 0, & 0 < \alpha < 2\pi \\ 1, & \alpha = 0 \end{cases}.$$

Thus $\{\mu_{n,\omega}\}$ is an ergodic sequence. Now let S be a p-set of integers with $p > 0$, and let $\{k_1, k_2, \cdots\} = S \cap Z^+$. We shall call the sequence $\{k_1, k_2, \cdots\}$ a p-sequence. Note that the asymptotic density of a p-sequence is p, since it is easy to verify that the sequence of measures $\{\nu_n\}$, where ν_n puts mass $1/n$ on the first n integers is an ergodic sequence. Now we have

$$\lim_n \int_Z \chi_S(j)\mu_{n,\omega}(d_j) = \lim_n \frac{1}{n}\sum_{j=1}^n f(T^j\omega)\chi_S(j) = p$$

since S is a p-set, and hence

$$\lim_n \frac{1}{k_n}\sum_{j=1}^n f(T^{k_j}\omega) = p.$$

Finally then

$$\lim_n \frac{1}{n}\sum_{j=1}^n f(T^{k_j}\omega) = \lim_n \frac{k_n}{n}\frac{1}{k_n}\sum_{j=1}^n f(T^{k_j}\omega) = 1,$$

since $\lim_n k_n/n = 1/p$. We summarize in the

THEOREM. *Let T be weakly mixing and let $f \in L_1(P)$. There exists a set Ω' of probability one such that if $\omega \in \Omega'$ we have*

$$\lim_n \frac{1}{n}\sum_{j=1}^n f(T^{k_j}\omega) = \int_\Omega f(\omega)P(d\omega)$$

for every p-sequence $\{k_1, k_2, \cdots\}$ with $p > 0$.

REMARK. If B is a continuity set for Lebesgue measure on T and α is an irrational multiple of π, then as noted above, the sequence $\{k_1, k_2, \cdots\}$ is an $l(B)$-sequence, if k_j is the jth entry time of $e^{ik\alpha}$ into B. Such sequences were called uniform sequences by Brunel and Keane [2]. They showed that if $f \in L_1(P)$ and $\{k_n\}$ is a uniform sequence then

$$\lim_n \frac{1}{n} \sum_{j=1}^{n} f(T^{k_j}\omega) = \int_\Omega f(\omega)P(\omega) \quad \text{a.e.}$$

Our results are more general since not every p-sequence is necessarily a uniform sequence, and more importantly, our set Ω' of probability one does not depend on the particular p-sequence chosen. To see that not every p-sequence is a uniform sequence we shall refer to a remarkable example due to Katznelson [3]. In his paper Katznelson constructed a 0-sequence $\{k_1, k_2, \cdots.\}$ which is dense in $\bar{Z}$. It follows immediately that this sequence cannot be a uniform sequence. Thus if $\{k_1, k_2, \cdots\}$ is a p-sequence for $p > 0$ and we take the union of this p-sequence with Katznelson's sequence we obtain a new p-sequence which is not a uniform sequence.

0-sequences

As noted above our theorem holds for p-sequences with $p > 0$. In this section we show by an example that it does not hold in general when $p = 0$. We shall need a result which we have not seen in print, but we are told is known to harmonic analysts. Let $S = \{k_1, k_2, \cdots\}$ be a lacunary sequence, i.e. $k_{n+1}/k_n \geqq \rho > 1$, $n = 1, 2, \cdots$. Then we have

LEMMA. $m(\bar{S}) = 0$.

Since a closed set of measure zero is a continuity set for Haar measure it follows that every lacunary sequence is a 0-sequence. Now suppose T is weakly mixing but not strongly mixing. Let U be the associated unitary operator defined by $Uf(\omega) = f(T\omega)$, for $f \in L_2(P)$. Then we may choose $f \in L_2(P)$ with the following properties:

i) f is bounded,

ii) $\int_\Omega f(\omega)P(d\omega) = 0$,

iii) The spectral measure μ_f of f is continuous and singular with respect to Lebesgue measure.

The spectral measure μ_f is the measure on C with Fourier coefficients $\hat{\mu}_f(k) = (U^k f, f)$. The fact that such an f exists is an easy consequence of the fact that T is weakly mixing but not strongly mixing. Moreover we may choose f to be real. Then there exists a sequence $\{r_n\}$ such that $\hat{\mu}_f(r_n) \to 0$ and another sequence $\{j_n\}$ such that $\hat{\mu}_f(j_n) \geqq \varepsilon > 0$ for $n = 1, 2, \cdots$ for some $\varepsilon > 0$. This follows from the fact that on one hand μ is continuous and on the other hand T is not strongly mixing. Thus by appropriately combining the two sequences and choosing subsequences we can find a sequence $\{k_n\}$ which is lacunary and such that $(1/n)\Sigma_{j=1}^{n}(U^{k_j}f, f)$ oscillates infinitely often from above $3\varepsilon/4$ to below $\varepsilon/4$. But then it follows from the bounded convergence theorem that $(1/n)\Sigma_{j=1}^{n} f(T^{k_j}\omega)$ cannot converge a.e.

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